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VALUE DISTRIBUTION AND UNIQUENESS OF CERTAIN TYPES OF DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we investigate the value distribution and uniqueness problem of q-shift difference polynomials sharing a small function. With the notion of weakly weighted sharing and relaxed weighted sharing we extend some well known previous results.

1. Introduction, Definitions and Results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let k be a positive integer or infinity and $a \in C \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than k. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f), m(r, f), N(r, f), \overline{N}(r, f), S(r, f)$ and so on, that can be found, for instance, in [4], [12]. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for zeros of $f - a$ with multiplicity less or equal to k, and by $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for zeros of $f - a$ with multiplicity atleast k and $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

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Let $N_E(r,a;f,g)(\overline{N}_E(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)(\overline{N}_0(r,a;f,g))$ the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

We now explain in the following definition the notion of weakly weighted sharing which was introduced by Lin and Lin [8].

Definition 1 ([8]). Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}^E_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^O_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g, and both of their multiplicities are not less than k.

Definition 2 ([8]). For $a\in C\cup\{\infty\}$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)-\overline{N}^E_{k)}(r,a;f,g)=S(r,f),$$

$$\overline{N}_{k)}\left(r,\frac{1}{g-a}\right)-\overline{N}^E_{k)}(r,a;f,g)=S(r,g),$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{f-a}\right)-\overline{N}^O_{(k+1}(r,a;f,g)=S(r,f),$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{g-a}\right)-\overline{N}^O_{(k+1}(r,a;f,g)=S(r,g),$$

or if $k=0$ and

$\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_0(r,a;f,g)=S(r,f),\overline{N}\left(r,\frac{1}{g-a}\right)-\overline{N}_0(r,a;f,g)=S(r,g),$
then we say f and g weakly share a with weight k. Here we write f,g share "(a,k)"

to mean that f, g weakly share a with weight k.

Now it is clear from Definition 2 that weakly weighted sharing is a scaling between IM and CM.

Recently, A. Banerjee and S. Mukherjee [1] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

Definition 3 ([1]). We denote by $\overline{N}(r, a; f \mid= p; g \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 4 ([1]). Let f, g share a "IM". Also let k be a positive integer or ∞ and $a \in C \cup \{\infty\}$. If

$$\sum_{p,q \leq K} \overline{N}(r, a; f \mid= p; g \mid= q) = S(r),$$

then we say f and g share a with weight k in a relaxed manner. Here we write f and g share $(a, k)^*$ to mean that f and g share a with weight k in a relaxed manner.

W. K. Hayman proposed the following well-known conjecture in [5].

Hayman's conjecture. If an entire function f satisfies $f^n f' \neq 1$ for all positive integers $n \in N$, then f is a constant.

It has been verified by Hayman himself in [6] for the case $n > 1$ and Clunie in [3] for the case $n \geq 1$, respectively.

It is well-known that if f and g share four distinct values CM, then f is Mobius transformation of g. In 2011, Liu and Cao [10], have obtained results on the uniqueness and value distribution of q-shift difference polynomials. Some of them are stated below.

Theorem A. [[10], Theorem 1.1] Let $f(z)$ be a transcendental meromorphic (resp. entire) function with zero order, and let m, n be positive integers and a, q be non-zero complex constants. If $n \geq 6$ (resp. $n \geq 2$), then $f^n(z)(f^m(z) - a)f(qz + c) - \alpha(z)$

has infinitely many zeros, where $\alpha(z)$ is a non-zero small function with respect to f. In particular, if $f(z)$ is a transcendental entire function and $\alpha(z)$ is a non-zero rational function, then m and n can be any positive integers.

Theorem B. [[10], Theorem 1.5] Let $f(z)$ and $g(z)$ be a transcendental entire functions with zero order. If $n \geq m+5$, and $f^n(z)(f^m(z)-a)f(qz+c)$ and $g^n(z)(g^m(z)-a)g(qz+c)$ share a non-zero polynomial $p(z)$ CM, then $f(z) \equiv g(z)$.

In 2015, On the basis of Theorems A and B, Q. Zhao and J. Zhang [14] study the k-th derivative of q-shift difference polynomials and proved the following results.

Theorem C. Let $f(z)$ be a transcendental meromorphic function with zero order, and let n, k be positive integers. If $n > k+5$, then $(f^n(z)f(qz+c))^{(k)} - 1$ has infinitely many zeros.

Theorem D. Let $f(z)$ be a transcendental entire function with zero order, and let n, k be positive integers, then $(f^n(z)f(qz+c))^{(k)} - 1$ has infinitely many zeros.

Theorem E. Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integer. If $n > 2k+5$, and $(f^n(z)f(qz+c))^{(k)}$ and $(g^n(z)g(qz+c))^{(k)}$ share z CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.

Theorem F. Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integer. If $n > 2k+5$, and $(f^n(z)f(qz+c))^{(k)}$ and $(g^n(z)g(qz+c))^{(k)}$ share 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.

When sharing a single value IM, and obtain the following theorems.

Theorem G. Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integer. If $n > 5k+11$, and $(f^n(z)f(qz+c))^{(k)}$ and $(g^n(z)g(qz+c))^{(k)}$ share a value z IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.

Theorem H. Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integer. If $n > 5k + 11$, and $(f^n(z)f(qz+c))^{(k)}$ and $(g^n(z)g(qz+c))^{(k)}$ share 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.

In this paper by introducing the small function $\alpha(z)$, we prove the following results.

Theorem 1. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 4k + m + \sigma + 5$ is an integer. $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ share "$(\alpha(z), 2)$", then $f(z) = g(z)$.

Theorem 2. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 6k + 3m + 2\sigma + 6$ is an integer. If $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ share $(\alpha(z), 2)^*$, then $f(z) = g(z)$.

without the notions of weakly weighted sharing and relaxed weighted sharing we prove the following theorem.

Theorem 3. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 10k + 5m + +5\sigma + 7$ is an integer. If $\overline{E}_{2)}\left(\alpha(z), \left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]\right) = \overline{E}_{2)}\left(\alpha(z), \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right.\right.$ then $f(z) \equiv g(z)$.

2. **Lemmas**

In this section, we present some lemmas which play an important role in the proof of the main results. We will denote by H the following function;

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

Lemma 1 ([1]). Let H be defined as above. If F and G share "(1, 2)" and $H \not\equiv 0$, then

$$\begin{aligned} T(r,F) \leq\ & N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) \\ & - \sum_{p=3}^{\infty} \overline{N}_{(p}\left(r,\frac{G}{G'}\right) + S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2 ([1]). Let H be defined as above. If F and G share $(1,2)^*$ and $H \not\equiv 0$, then

$$\begin{aligned} T(r,F) \leq\ & N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \overline{N}\left(r,\frac{1}{F}\right) \\ & + \overline{N}(r,F) - m\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 3 ([13]). Let H be defined as above. If $H \equiv 0$ and

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G)}{T(r)} < 1, \ r \in I,$$

where $T(r) = \max\{T(r,F), T(r,G)\}$ and I is a set with infinite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 4 ([2]). Let $f(z)$ be a meromorphic function in the complex plane of finite order $\sigma(f)$, and let η be a fixed non-zero complex number. Then for each $\epsilon > 0$, one has

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma(f)-1+\epsilon}) + O(logr)$$

Lemma 5 ([11]). Let $f(z)$ be an entire function of finite order $\sigma(f)$, c a fixed non-zero complex number, and

$$P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + ... + a_1 f(z) + a_0$$

where $a_j (j = 0, 1, ..., n)$ are constants. If $F(z) = P(z)f(z+c)$, then

$$T(r, F) = (n+1)T(r, f) + O(r^{\sigma(f)-1+\epsilon}) + O(logr).$$

Lemma 6 ([9]). Let F and G be two nonconstant entire functions, and $p \geq 2$ an integer. If $\overline{E}_{p)}(1, F) = \overline{E}_{p)}(1, G)$ and $H \not\equiv 0$, then

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, F) + S(r, G).$$

Lemma 7 ([7]). Let $f(z)$ be a nonconstant meromorphic function, and let s, k be two positive integers. Then

$$N_s(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{(s+k)}(r, \frac{1}{f}) + S(r, f),$$
$$N_s(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Clearly, $\overline{N}(r, \frac{1}{f^{(k)}}) = N_1(r, \frac{1}{f^{(k)}})$.

Lemma 8. Let f and g be two entire functions, suppose that $c_j (j = 1, 2, \cdots, d)$ are nonzero complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \geq 1$ and $k (\geq 0)$ are integers and let $F = \left[f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $G = \left[g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$. If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, c_2; G) = \overline{N}(r, 0; F)$, then $n \leq 2k + m + \sigma + 2$.

Proof. We put $F_1 = f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $G_1 = g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g(z+c_j)^{v_j}$, by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, c_1; F) + S(r, F) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) \end{aligned} \tag{2.1}$$

Using equation (2.1), in Lemmas 2 and 4, we obtain

$$
\begin{aligned}
(n+m+\sigma)T(r,f) &\le T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;F_1)+S(r,f)\\
&\le \overline{N}(r,0;G)+N_{k+1}(r,0;F_1)+S(r,f)\\
&\le N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\
&\le (k+m+\sigma+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned} \tag{2.2}
$$

Similarly,

$$(n+m+\sigma)T(r,g) \le (k+m+\sigma+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$(n-2k-m-\sigma-2)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g).$$

Which gives $n \le 2k+m+\sigma+2$.

This proves the lemma.

Lemma 9. Suppose that f and g are two transcendental entire function of finite order, Suppose that $c_j(j=1,2,\cdots,d)$ are nonzero complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m \ge 1$ and $k(\ge 0)$ are integers. If $n \ge m+5$ and $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^d f(z+c_j)^{v_j}\right]^{(k)} = \left[g^n(z)(g^m(z)-1)\prod_{j=1}^d g(z+c_j)^{v_j}\right]^{(k)}$ then $f=tg$ where $t^m=1$.

3. Proof of Theorem 1

Let

$$F(z)=\frac{\left[f^n(z)(f^m(z)-1)\prod_{j=1}^d f(z+c_j)^{v_j}\right]^{(k)}}{\alpha(z)},\ G(z)=\frac{\left[g^n(z)(g^m(z)-1)\prod_{j=1}^d g(z+c_j)^{v_j}\right]^{(k)}}{\alpha(z)}$$

Then $F(z)$ and $G(z)$ share "(1, 2)" except the zeros or poles of $\alpha(z)$. By Lemma 5, we have

$$T(r,F(z)) = T(r,f^n(f^m-1)\prod_{j=1}^d f(z+c_j)^{v_j})+k\overline{N}(r,f)+S(r,f), \tag{3.1}$$

$$T(r,G(z)) = T(r,g^n(g^m-1)\prod_{j=1}^d g(z+c_j)^{v_j})+k\overline{N}(r,\frac{1}{f})+S(r,g). \tag{3.2}$$

Also from Lemma 7, we obtain

(3.3)

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right) &\le N_{k+2}\left(r,\frac{1}{f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}}\right)+S(r,f)\\ &\le (k+2)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^m-1}\right)+N\left(r,\frac{1}{\prod_{j=1}^{d}f(z+c_j)^{v_j}}\right)\\ &+k\overline{N}(r,f)+S(r,f)\\ &\le (2k+m+\sigma+2)T(r,f)+S(r,f)\end{aligned}$$

and

$$N_2\left(r,\frac{1}{G}\right)\le(2k+m+\sigma+2)T(r,g)+S(r,g) \tag{3.4}$$

Suppose $H\not\equiv 0$, then by Lemma 1 and Lemma 4, we have

(3.5)

$$T(r,F)+T(r,G)\le 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g)$$

$$(n+m+1)[T(r,f)+T(r,g)]\le(4k+2m+2\sigma+4)[T(r,f)+T(r,g)]+S(r,f)+S(r,g)$$

$$(n-4k-m-2\sigma-3)[T(r,f)+T(r,g)]\le O(r^{\sigma(f)-1+\epsilon})+O(r^{\sigma(g)-1+\epsilon})+S(r,f)+S(r,g)$$

which contradicts with $n\ge 4k+m+2\sigma+3$. Thus we have $H\equiv 0$. Note that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)&\le(2k+m+\sigma+1)T(r,f)+(2k+m+\sigma+1)T(r,g)\\ &+S(r,f)+S(r,g)\\ &\le T(r).\end{aligned}$$

where $T(r)=\max\{T(r,F),T(r,G)\}$. By Lemma 3, we deduce that either $FG=1$ or $F=G$

Let $FG=1$. Then,

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}\cdot\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}=\alpha^2$$

$$\left[f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+\cdots+1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}\cdot\left[g^n(z)(g(z)-1)(g^{m-1}(z)+g^{m-2}(z)+\right.$$

It can be easily viewed from above that

$$N(r,0;f)=S(r,f) \text{ and } N(r,1;f)=S(r,f)$$

Thus,

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

Which is not possible. Therefore, we must have $F=G$, and then

$$\Big[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\Big]^{(k)}=\Big[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big]^{(k)}$$

Integrating above, we get,

$$\Big[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\Big]^{(k-1)}=\Big[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big]^{(k-1)}+C_{k-1}$$

Where C_{k-1} is a constant. If $C_{k-1}\neq 0$, using Lemma 8, it follows that $n\leq 2k+m+\sigma$ a contradiction. Hence $C_{k-1}=0$, repeating k times, we deduce that,

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}=g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}$$

which by Lemma 9, gives $f=tg$ where t is a constant satisfying $t^m=1$. This proves Theorem 1.

4. **Proof of Theorem 2**

Let

$$F(z)=\frac{[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}]^{(k)}}{\alpha(z)},G(z)=\frac{[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}]^{(k)}}{\alpha(z)}$$

Then $F(z)$ and $G(z)$ share $(1,2)^*$ except the zeros or poles of $\alpha(z)$. Obviously

(4.1)

$$2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G)$$
$$\leq(6k+3m+2\sigma+6)T(r,f)+(6k+3m+2\sigma+6)T(r,g)+S(r,f)+S(r,g).$$

According to 4.1 and Lemma 2, we can prove Theorem 2 in a similar way as in Section 3.

5. **Proof of Theorem 3**

Let

$$F(z)=\frac{[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}]^{(k)}}{\alpha(z)},G(z)=\frac{[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}]^{(k)}}{\alpha(z)}$$

Then $\overline{E}_{2)}(1,[f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}]^{(k)})=\overline{E}_{2)}(1,[g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}]^{(k)})$ except the zeros or poles of $\alpha(z)$. Obviously

(5.1)

$$2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+3\overline{N}\left(r,\frac{1}{F}\right)+3\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G)$$
$$\leq(10k+5m+5\sigma+7)T(r,f)+(14k+5m+5\sigma+7)T(r,g)+S(r,f)+S(r,g).$$

Using 5.1 and Lemma 6, we can prove Theorem 3 in a similar way as in Section 3.

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